

GENERAL MECHANICS

NON-CLASSICAL THERMOELASTICITY*

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ABSTRACT. The aim of the present study is to offer new, non-classical theory of thermoelasticity, which is able to solve the paradox of the infinite speed of propagation of the thermal perturbation, resulting from the application of Fourier law for the heat conduction. In this article we offer an extension of Clausius-Duhem inequality, which consists in replacement of the momentary dependence between the entropy and energy fluxes and their sources with dependences of memory type. As a result we found the restrictions, following from the validity of the second law of thermodynamics, on the constitutive equations and obtained hyperbolic differential equation for the caloric balance.

KEY WORDS: extended, non-classical, thermoelasticity, Clausius-Duhem, memory, paradox, infinite speed, heat conduction.

I. Introduction

The study of the effect of the heat stress for deformable solids started at 1838 with Duhamel articles [1, 2], where he introduced the so called thermal strains. Similar approach was used by Neumann [3] forty seven years later. The main offset of these two theories is their not acceptance of the deformation influence on the thermal conduction. Effort in this direction was made by Voigt [4] and also by Jeffreys [5], but they used the equilibrium thermodynamics and consequently their theories are limited for quasi equilibrium processes.

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The classical theory of the thermo-elasticity, based on the irreversible thermodynamics of de Groot [6], was offered by Biot [7]. However the theory of Biot is founded on the Fourier [8] law for heat conduction and therefore allows for infinite speed of propagation of thermal perturbations, which is physically unrealistic. An attempt, to clean this paradox, was done by Green and Lindsay [9] in their model for thermo-elastic media, whose heat flux h_K depends on the rate of the absolute temperature θ :

$$(1) \quad h_K = b_K \frac{\partial \theta}{\partial t} - \lambda_{KL} \theta_{,L},$$

where b_K is an anti-symmetric vector and λ_{KL} is the classical heat conductivity tensor. This theory, however, is with narrow application because of the fact that an anti-symmetric vector vanishes for all cases of central material symmetry. Recently the authors of the present article, in their study [10], pointed out that the paradox of infinite speed of propagation of thermal perturbation is associated with the momentary relationship between entropy and energy flux, in Clausius-Duhem inequality, neglecting any relaxation processes between entropy and energy flux. Instead, it is suggested in [10] the more general relationship of memory type. Therefore, they are able to derive new extended formulation for the heat conduction, which represents a generalization of the model, offered by Maxwell [11]

$$(2) \quad h_K + \tau_r \frac{\partial h_K}{\partial t} = -\lambda_{KL} \theta_{,L},$$

and the model offered by Green and Lindsay, (1).

The aim of the present study is to offer strict and general approach for the constitutive theory of the thermoelastic continua, leading to hyperbolic partial differential equation for caloric balance, instead of the parabolic one and in such way to exclude the paradox connected with the infinite speed of propagation of the thermal perturbation.

II. Kinematics

A material point X with rectangular coordinates $X_K \in B$ at $t = 0$ is carried to position $x \in b$ with rectangular coordinates x_k at time t . The kinematics of the body is represented by the continuous and isomorphic transformations

$$(3) \quad x_k = \chi_k(X_K, t), \quad X_K = \bar{\chi}_K(x_k, t).$$

By definition

$$(4) \quad x_{k,K} = \frac{\partial}{\partial X_K} \chi_k(X_K, t)|_t \quad \text{and} \quad X_{K,k} = \frac{\partial}{\partial x_k} \bar{\chi}_K(x_k, t)|_t$$

are the deformation gradients and

$$(5) \quad v_k = \frac{\partial}{\partial t} \chi_k(X_K, t)|_{X_K}$$

is the velocity field.

The material derivative $\frac{D}{Dt}$ is defined by

$$(6) \quad \frac{D}{Dt} \varphi = \frac{\partial}{\partial t} \varphi(X_K, t)|_{X_K},$$

where $\varphi(X_K, t)$ is continuous function with respect to X_K and t .

Eulerian and lagrangian strains are represented by

$$(7) \quad e_{kl} = X_{K,k} X_{K,l} - \delta_{kl} \quad \text{and} \quad E_{KL} = x_{k,K} x_{k,L} - \delta_{KL},$$

where δ_{kl} , δ_{KL} are Kroniker symbols.

Under deformation, the volume $dV \in B$ is transformed in $dv \in b$

$$(8) \quad dv = j dV, \quad j = \det\{x_{k,K}\}.$$

III. Conservation principles

Mass conservation

$$(9) \quad \rho_0 = j \rho,$$

where ρ_0 and ρ are the initial and momentary mass densities;

Moment conservation

$$(10) \quad \rho_0 \frac{D}{Dt} v_k = T_{Kk,K} + \rho_0 f_k,$$

where the first Piola stress tensor T_{Kk} related to the Cauchy stress tensor t_{kl} as

$$(11) \quad T_{Kl} = j X_{K,l} t_{kl},$$

and f_k is the field of the external forces;

Moment of momentum conservation

$$(12) \quad T_{KL} = T_{LK},$$

where the second Piola stress tensor T_{KL} related to the Cauchy stress tensor t_{kl} as

$$(13) \quad T_{KL} = j X_{K,k} X_{K,l} t_{kl};$$

Energy conservation

$$(14) \quad \rho_0 \frac{D}{Dt} e = T_{KL} \frac{D}{Dt} E_{KL} + h_{K,K} + \rho_0 e^*,$$

where e and e^* are the energy density and the density of energy sources. The material heat flux h_K related to the spatial h_k as

$$(15) \quad h_K = j X_{K,k} h_k;$$

Dissipation principle

According to the Clausius-Duhem formulation of the Second Law of Thermodynamic, the rate of the entropy η satisfies the balance equation

$$(16) \quad \rho_0 \frac{d\eta}{dt} = J_{K,K}^\eta + \rho_0 \eta^* + \rho_0 \hat{\eta},$$

where

i) the entropy flux J_K^η is proportional to the heat flux

$$(17) \quad J_K^\eta = \alpha h_K,$$

ii) the rate of the entropy sources is proportional to the rate of the heat sources

$$(18) \quad \eta^* = \alpha e^*,$$

iii) the internal source of the entropy due to the local dissipation is non-negative

$$(19) \quad \hat{\eta} \geq 0,$$

iv) the reciprocal value of α in (17, 18), by definition, is the absolute temperature

$$(20) \quad \theta = \frac{1}{\alpha}.$$

The requirements (17-20) lead to the well known formulation of the Clausius-Duhem inequality

$$(21) \quad \rho_0 \frac{d\eta}{dt} - \left(\frac{h_K}{\theta} \right)_{,K} - \rho_0 \frac{e^*}{\theta} \geq 0.$$

Something of importance in the above inequality is the explicit acceptance of momentary dependences between the entropy flux and the heat flux (17) and the same between the rate of entropy sources and rate of energy sources (18). The wide interpretation of entropy is as a measure for the structural order, degradation and inhomogeneity. However, there are many experimental and

theoretical evidences, which suggest non momentary dependence between the energy supply and structural order of the systems. It is a reason the Clausius-Duhem inequality (21) to be extended by replacement of (17, 18) with the memory type relations

$$(22) \quad J_K^\eta(t) = \int_{-\infty}^t \frac{\beta(t-t')}{\theta(t)} h_K(t') dt', \quad \eta^*(t) = \int_{-\infty}^t \frac{\beta(t-t')}{\theta(t)} e^*(t') dt',$$

where the kernel $\beta(t-t')$ is normalized memory function

$$(23) \quad \int_{-\infty}^t \beta(t-t') dt' = 1.$$

The simplest form of memory dependence is the case of strongly fading memory – rate dependence. In this case the kernel $\beta(t-t')$ is approximated by the asymmetric Dirac function and its derivative as

$$(24) \quad \beta(t-t') \approx \delta_+(t-t') + \tau^{ch} \frac{d}{dt'} \delta_+(t-t'),$$

where we call τ^{ch} – “chaos relaxation time”.

The next, similar to (21) extension of Clausius-Duhem inequality, follows from (16, 19, 22, 24):

$$(25) \quad \rho_0 \frac{d\eta}{dt} - \left(\frac{H_K}{\theta} \right)_{,K} - \rho \left(\frac{\varepsilon^*}{\theta} \right) \geq 0,$$

where

$$(26) \quad H_K = h_K + \tau^{ch} \frac{D}{Dt} h_K, \quad \varepsilon^* = e^* + \tau^{ch} \frac{D}{Dt} e^*.$$

IV. Constitutive equations

By substitution of (14) and (26) into (25) we obtain

$$(27) \quad \rho_0 \frac{D}{Dt} \Psi - T_{KL} \frac{D}{Dt} E_{KL} + \rho_0 \eta \frac{D\theta}{Dt} - H_K \frac{\theta_{,K}}{\theta} \leq 0,$$

where

$$(28) \quad \Psi = e + \tau^{ch} \left(\frac{De}{Dt} - \frac{1}{\rho_0} T_{KL} \frac{D}{Dt} E_{KL} \right) - \theta \eta,$$

by definition, is the extended free energy density. If the chaos relaxation time τ^{ch} is accepted as infinitesimal, then the extended free energy formulation coincides with the classical one.

As thermodynamic state parameters we assume: the temperature θ , the temperature gradient $\theta_{,K}$, their rates $\frac{D\theta}{Dt}$, $\frac{D\theta_{,K}}{Dt}$, and the strain tensor E_{KL} . So for the constitutive equations we have

$$(29) \quad \Psi = \Psi \left(\theta, \theta_{,K}, E_{KL}, \frac{D\theta}{Dt}, \frac{D\theta_{,K}}{Dt} \right),$$

$$(30) \quad \eta = \eta \left(\theta, \theta_{,K}, E_{KL}, \frac{D\theta}{Dt}, \frac{D\theta_{,K}}{Dt} \right),$$

$$(31) \quad T_{MN} = T_{MN} \left(\theta, \theta_{,K}, E_{KL}, \frac{D\theta}{Dt}, \frac{D\theta_{,K}}{Dt} \right),$$

$$(32) \quad H_K = H_K \left(\theta, \theta_{,K}, E_{KL}, \frac{D\theta}{Dt}, \frac{D\theta_{,K}}{Dt} \right).$$

Taking into account (29 – 32) in (27) we obtain

$$(33) \quad \rho_0 \left(\frac{\partial \Psi}{\partial \theta} + \eta \right) \frac{D\theta}{Dt} + \left(\rho_0 \frac{\partial \Psi}{\partial E_{KL}} - T_{KL} \right) \frac{DE_{KL}}{Dt} + \rho_0 \frac{\partial \Psi}{\partial \frac{D\theta}{Dt}} \frac{D^2\theta}{Dt^2} +$$

$$\rho_0 \frac{\partial \Psi}{\partial \theta_{,K}} \frac{D\theta_{,K}}{Dt} + \rho \frac{\partial \Psi}{\partial \frac{D\theta_{,K}}{Dt}} \frac{D^2\theta_{,K}}{Dt^2} - \frac{\theta_{,K}}{\theta} H_K \leq 0.$$

The necessary and sufficient conditions for the validity of the inequality (33), under the assumptions (22–24) for arbitrary values of the state parameters, are:

$$(34) \quad \rho_0 \left(\frac{\partial \Psi}{\partial \theta} + \eta \right) \frac{D\theta}{Dt} + \left(\rho_0 \frac{\partial \Psi}{\partial E_{KL}} - T_{KL} \right) \frac{DE_{KL}}{Dt} - H_K \frac{\theta_{,K}}{\theta} \leq 0,$$

$$(35) \quad \frac{\partial \Psi}{\partial \theta_{,K}} = \frac{\partial \Psi}{\partial \frac{D\theta}{Dt}} = \frac{\partial \Psi}{\partial \frac{D\theta_{,K}}{Dt}} = 0,$$

$$(36) \quad T_{KL} = \rho_0 \frac{\partial \Psi}{\partial E_{KL}},$$

$$(37) \quad \rho_0 \left(\frac{\partial \Psi}{\partial \theta} + \eta \right) \frac{D\theta}{Dt} - H_K \frac{\theta_{,K}}{\theta} \leq 0.$$

The sufficient conditions the inequality (37) to be satisfied is:

$$(38) \quad \begin{pmatrix} \rho_0 \left(\frac{\partial \Psi}{\partial \theta} + \eta \right) \\ -\frac{\{H_K\}}{\theta} \end{pmatrix} = \begin{pmatrix} -B & -\frac{\{b_K\}}{\theta} \\ -\frac{\{b_K\}}{\theta} & \frac{\{\lambda_{KL}\}}{\theta} \end{pmatrix} \begin{pmatrix} \frac{D\theta}{Dt} \\ \{\theta_{,L}\} \end{pmatrix},$$

where

$$(39) \quad \begin{pmatrix} -B & -\frac{\{b_K\}}{\theta} \\ -\frac{\{b_K\}}{\theta} & +\frac{\{\lambda_{KL}\}}{\theta} \end{pmatrix}$$

is non-positive matrix. The anti-symmetric vector b_K is equal to zero for all cases of central material symmetry.

V. Linear constitutive equations

With the help of (26, 35–38), for the linear variant of the present theory, we obtain:

$$(40) \quad \Psi = \frac{1}{2}\alpha(\theta - \theta_0)^2 + \frac{1}{2}\beta_{KLMN}\varepsilon_{KL}\varepsilon_{MN} + \chi_{KL}(\theta - \theta_0)\varepsilon_{KL},$$

$$(41) \quad \eta = -\alpha(\theta - \theta_0) - B\frac{\partial \theta}{\partial t} - \frac{1}{\rho_0\theta_0}b_K\theta_{,K} - \chi_{KL}\varepsilon_{KL},$$

$$(42) \quad T_{KL} = \beta_{KLMN}\varepsilon_{MN} + \chi_{KL}(\theta - \theta_0),$$

$$(43) \quad h_K + \tau^{ch}\frac{\partial}{\partial t}h_K = b_K\frac{\partial \theta}{\partial t} - \lambda_{KL}\theta_{,L}.$$

VI. Balance of the caloric energy

The next non-classical balance equation, for the caloric balance, follows from equations (14, 28, 41-43):

$$(44) \quad \frac{\partial^2 \theta}{\partial t^2} + \frac{\alpha}{B}\frac{\partial}{\partial t}\theta = \frac{\lambda_{KL}}{\rho\theta_0 B}\theta_{,LK} - \frac{1}{\theta_0 B}\left(e^* + \tau^{ch}\frac{\partial}{\partial t}e^*\right) - \frac{1}{\theta_0 B}\chi_{KL}\frac{\partial}{\partial t}E_{KL}.$$

VII. Conclusions

In the present study we develop strict thermodynamic theory for thermoelastic continuum by use of extension of the Clausius-Duhem inequality by

introducing memory functional relationship between the entropy flux and heat flux and their sources, which take into account the relaxation of the “chaotic processes”. This basic assumption gives us the ability to formulate a general theory of the coupled thermoelastic and heat conduction phenomenon, which yields to finite velocity of propagation of thermal perturbation. The obtained equation for the heat conduction represents a generalization of the models offered by Maxwell [11] and Green and Lindsay [9]. However, as it is seen in (44) the proposed in [9] new non classical term $-b_K \frac{\partial \theta}{\partial t}$, for heat conduction equation (43), does not affect the caloric balance.

As conclusion we would say that the proposed formulation of the Second Law of thermodynamics [10] would be explored in other problems of the continuum physics, beyond the coupled thermoelasticity, as well.

REFERENCES

- [1] DUHAMEL, J. Some Memoire sur les Phenomenes Thermo-Mechanique. *J. de L Ecole Polytech.*, **15** (1837), 1–15.
- [2] DUHAMEL, J. Mémoire sur le Calcul des Actions Moleculaires Developpees par les Changements de Temperature dans les Corps Solides. *Mem. par Div. Say. a l'Acad. Roy. des Sci. de l'Inst. de France*, (1838), 440–498.
- [3] NEUMANN, F. Vorlesungen Uber die Theorie der Elasticitat, Brestau, Meyer, 1885.
- [4] VOIGT, W. Lehrbuck der Kristallphysik, Teubner, 1910.
- [5] JEFFREYS, H. The Thermodynamics of an Elastic Solid. *Proc. Camb. Phil. Soc.*, **26** (1930), 101–106.
- [6] DE GROOT, S. R. Thermodynamics of Irreversible Processes, Amsterdam, 1952.
- [7] BIOT, M. Thermoelasticity and Irreversible Thermodynamics. *J. Appl. Phys.*, **27** (1956) 249–253.
- [8] FOURIER, J. Théorie Analytique de la Chaleur, Paris, 1822.
- [9] GREEN, A. E., K. A. LINDSAY. Thermoelasticity. *J. Elasticity*, **2** (1971) 1–7.
- [10] PETROV, N., A. SZEKERES. New Approach to the Non-Classical Heat Conduction. *Journal of Theoretical and Applied Mechanics*, Sofia, **38** (2008), No. **3**, 61–70.
- [11] MAXWEL, J. On the Dynamical Theory of Gases. *Phil. Trans. Royal Soc. London*, **157** (1867), 49–88.