Journal of Theoretical and Applied Mechanics, Sofia,
2008, vol. 38, No. 3, pp 61-70

NEW APPROACH TO THE NON-CLASSICAL HEAT CONDUCTION

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[Received 17 March 2008. Accepted 09 June 2008]

Abstract. The aim of the present study is to offer a non-classical model able to solve the problem connected with the paradox of the infinite speed of propagation of the thermal perturbation. The principle assumption is that the momentary dependence between the entropy and heat fluxes in Clausius – Duhem inequality does not take into account the relaxation phenomena due to the micro-structural formation and degradation processes. We reformulate this dependence as memory type. As a result a hyperbolic partial differential equation for the caloric balance is obtained instead of parabolic one, by which the above mention paradox is eliminated. Also we obtain a new heat conduction equation which is a generalization of the equations offered by Maxwell, and Green and Lindsay.

Key words: non classical, heat, conduction, claussius, duhem, thermodynamics, caloric, hyperbolic, parabolic

1. Introduction

A fundamental drawback of the classical theory of heat conduction by Fourier law [1] is that it allows for infinite speed of propagation of thermal perturbations, which is physically unrealistic. This paradox has been first discussed by Maxwell [2] who offered the heat conduction equation:

(1) $$h_k + \tau_r \frac{\partial h_k}{\partial t} = -\lambda_{kl}\theta_{,l} \quad ,$$

where τ_r is the thermal relaxation time, θ is the absolute temperature, λ_{kl} is the heat conductivity tensor and h_k is the heat flux. Nowadays similar generalizations have been proposed in [3], [4], [5], etc.

The paradox of infinite speed of propagation has been removed also in [6] by model of thermo-elastic media, whose heat flux depends on the rate of the absolute temperature:

(2) $$h_k = b_k \frac{\partial \theta}{\partial t} - \lambda_{kl}\theta_{,l}$$

where b_k is an anti-symmetric vector.

The equations (1, 2) are example of the rate depending constitutive equations, which are a special case in the general theory of the fading memory [7-12]. In these studies Coleman et al. represent free energy density ψ and entropy density η as functions of the present values of the temperature gradients, but not of the gradient history [10, 11]. They state, that if history of the temperature gradient is considered as an independent variable, the principle of local action is violated. Hence they neglect the history of the temperature gradient in the proposed constitutive equations [8, 9, 12]. With this limitation of their theory they exclude the non-classical generalization of the heat conduction equation, admitting a finite speed of heat propagation. It was demonstrated in [13] that the acceptance for weak non-locality in the above sense gives possibility for generalization of the classical heat conduction models and to formulation of a phenomenological model which produces as particular cases all known linear heat conduction models. However this finding is not a resolution of the problem, because from physical point of view the finite speed of propagation of the thermal perturbation should be true in the case of local action as well, as for the case of non-local one. The heat conduction equation (1) was derived, also in [14, 15] on the basis of heuristic assumptions and analytical concern on the fundamental principles of the continuum mechanics [16].

2. Etymology of the idea for entropy in the continuum mechanics and new reformulation of the Clausius - Duhem inequality

The originator of the idea of "Entropy" is Clausius, who introduced the inequality [17, 18]:

(3) $$d\,S \geq \frac{\delta Q}{\theta} \quad ,$$

where δQ and $d\,S$ are the heat increment and the correspondent entropy increase.

The inequality (3) was reformulated by Duhem in 1911 [19] for continuum as:

(4) $$\frac{d\,S}{d\,t} \geq \oint_A \frac{h_k}{\theta} d\,A_k \quad ,$$

which leads to the local formulation

(5) $$\rho \frac{d\,\eta}{d\,t} \geq \left(\frac{h_k}{\theta}\right)_{,k} \quad .$$

In (4) dA_k is the oriented surface element. The values ρ and η in (5) are the mass density and entropy density. Truesdell and Toupin added in (5) a bulk term [20]. The obtained presentation of the Second Law of Thermodynamics is known as Clausius – Duhem inequality:

(6) $$\rho \frac{d\eta}{dt} - \left(\frac{h_k}{\theta} \right)_{,k} - \rho \frac{e^*}{\theta} \geq 0 \qquad ,$$

where e^* is the density of the heat sources. The physical meaning of inequality (6) was discussed by Petrov in [21]. He started from the following general local formulation of the balance for the entropy density:

(7) $$\rho \frac{d\eta}{dt} = J^{\eta}_{k,k} + \rho\eta^* + \rho\hat{\eta} \quad ,$$

where J^{η}_{k} is the entropy flux and η^* is the density of the rate of entropy sources due to volume distributed heat supply. The variable $\hat{\eta}$ is the term representing the density of entropy sources due to the local dissipation.

One can see that Clausius – Duhem inequality (6) follows from (7) if we postulate:

1. The entropy flux is proportional to the heat flux

(8) $$J^{\eta}_{k} = \alpha h_k \qquad .$$

2. The intensity of the distributed entropy sources is proportional to the intensity of the distributed heat sources:

(9) $$\eta^* = \alpha e^* \qquad .$$

3. The constant of proportionality α is equal to the reciprocal value of

the absolute temperature

(10) $$\alpha = \frac{1}{\theta} \quad .$$

4. The internal source of the entropy due to the local dissipation is non-negative

(11) $$\hat{\eta} \geq 0 \quad .$$

In the Clausius formulation (3) the entropy increase is interpreted as increase of the chaos. He pointed out [17] that the ice melting is a classical example of entropy increasing, resulting in the desegregation of the molecules of the body of ice. One can see that in Clausius - Duhem formulation, the relation between the entropy increment and the heat income is represented by momentary dependence. However in many cases the system structure and the correspondingly the entropy depends not only on the heat supply but also on the rate or history of the heat income. An example in this direction is the degree of crystallization of liquids and liquid solutions.

3. New reformulation for the second law of thermodynamics

To extend the Clausius - Duhem formulation, instead of the momentary dependence in (8, 9), we offer memory type relationship, which gives possibility to the relaxation phenomena in the processes of structure formation and degradation to be taken into account. Following such line we offer the next generalization of (8, 9):

(12) $$J_k^{\eta}(t) = \int_{-\infty}^{t} \frac{\beta(t-t')}{\theta(t)} h_k(t')\, dt'$$

(13) $$\eta^*(t) = \int_{-\infty}^{t} \frac{\beta(t-t')}{\theta(t)} e^*(t')\, dt' \ ,$$

where the kernel $\beta(t-t')$ is normalized memory function -

(14) $$\int_{-\infty}^{t} \beta(t-t')\, dt' = 1 \qquad .$$

Now if we put into (12, 13) Dirac delta function $\delta_+(t-t')$ as a particular case of $\beta(t-t')$ the result will be the classical formulation.

4. Non-classical heat conduction

To demonstrate that the generalization (12, 13) is not empty case we will explore this formulation for modelling the phenomena of non-classical heat conduction.

To make the derivation easy we suppose that the system is characterized by strongly finding memory. In such a case the kernel in (12, 13) could be approximated by the asymmetric Dirac function and its derivative as:

(15) $$\beta(t-t') \approx \delta_+(t-t') + \tau^{ch}\frac{d}{dt'}\delta_+(t-t') \ ,$$

where the "chaos relaxation time" τ^{ch} is the relaxation time of the micro-structural formation–degradation process. So instead of the Clausius - Duhem inequality (6) we explore the next new non-classical formulation of the

<u>Second law of thermodynamics</u>

(16) $$\rho\frac{d\eta}{dt} - \left(\frac{h_k + \tau^{ch}\frac{d}{dt}h_k}{\theta}\right)_{,k} - \rho\left(\frac{e^* + \tau^{ch}\frac{d}{dt}e^*}{\theta}\right) \geq 0 \ .$$

The balance of energy is represented by the

<u>Energy conservation law</u>

(17) $$\rho\frac{de}{dt} = h_{k,k} + \rho e^* \quad .$$

One can obtain from (16) and (17) the following representation of the Second Law of thermodynamics:

(18) $$\rho\frac{d\Psi}{dt}+\rho\eta\frac{d\theta}{dt}-\frac{\theta_{,k}}{\theta}H_{,k}\leq 0 \quad ,$$

where

(19) $$\Psi=e+\tau^{ch}\frac{de}{dt}-\theta\eta \quad ,$$

(20) $$H_k=h_k+\tau^{ch}\frac{d}{dt}h_k$$

are extended formulations of the free energy density and the heat flux. In (19, 20), if the chaos relaxation time is negligibly small value, then the extended free energy and heat flux coincide with their classical definitions.

5. Constitutive equations for non-classical heat conducting media

As thermodynamic state parameters we assume the temperature - θ, the temperature gradient $\theta_{,k}$ and their rates - $\frac{d\theta}{dt}$, $\frac{d\theta_{,k}}{dt}$. So for the constitutive equations we have:

(21) $$\Psi = \Psi(\theta,\theta_{,k},\frac{d\theta}{dt},\frac{d\theta_{,k}}{dt}) \quad ,$$

(22) $$\eta = \eta(\theta,\theta_{,k},\frac{d\theta}{dt},\frac{d\theta_{,k}}{dt}) \quad ,$$

(23) $$H_k = H_k(\theta,\theta_{,k},\frac{d\theta}{dt},\frac{d\theta_{,k}}{dt}) \quad .$$

Taking into account (19 - 21) in (18) we obtain:

(24) $$\rho\left(\frac{\partial \Psi}{\partial \theta}+\eta\right)\frac{d\theta}{dt}+\rho\frac{\partial \Psi}{\partial \frac{d\theta}{dt}}\frac{d^2\theta}{dt^2}+\rho\frac{\partial \Psi}{\partial \theta_{,k}}\frac{d\theta_{,k}}{dt}$$

$$+\ \rho\frac{\partial \Psi}{\partial \frac{d\theta_{,k}}{dt}}\frac{d^2\theta_{,k}}{dt^2}-\frac{\theta_{,k}}{\theta}H_k\le 0\,.$$

The necessary and sufficient conditions for the validity of the inequality (24), under the assumptions done, are:

(25) $$\rho\left(\frac{\partial \Psi}{\partial \theta}+\eta\right)\frac{d\theta}{dt}-H_k\frac{\theta_{,k}}{\theta}\le 0 \qquad ,$$

(26) $$\frac{\partial \Psi}{\partial \theta_{,k}}=\frac{\partial \Psi}{\partial \frac{d\theta}{dt}}=\frac{\partial \Psi}{\partial \frac{d\theta_{,k}}{dt}}=0 \qquad .$$

Sufficient condition to be satisfied inequality (25) is the validity of the following two relations:

(27) $$\frac{\partial \Psi}{\partial \theta}+\eta=-B\frac{d\theta}{dt}+b_l\,\theta_{,l} \qquad ,$$

(28) $$H_k=b_k\frac{d\theta}{dt}-\lambda_{kl}\,\theta_{,l} \qquad ,$$

where the symmetric matrix: $\begin{pmatrix} B & -\{b_k\} \\ -\{b_k\} & \{\lambda_{kl}\} \end{pmatrix}$

is non-negative. It follows from this requirement that B is non-negative constant and λ_{kl} is non-negative symmetric matrix. The vector b_k change its sign if we permutate the coordinate indexes (anti-symmetric vector). Consequently b_k are equal to zero if we have central symmetry.

For the case of linear heat-conductive media we have:

(29) $$\Psi = \frac{1}{2} A(\theta - \theta_0)^2 \quad ,$$

(30) $$\eta = - A(\theta - \theta_0) - B\frac{d\theta}{dt} + b_l \theta_{,l} \quad ,$$

(31) $$h_k + \tau^{ch} \frac{d}{dt} h_k = b_k \frac{d\theta}{dt} - \lambda_{kl} \theta_{,l} \quad ,$$

where θ_0 is the reference temperature.

One can see that (31) is generalization of the heat conduction equations (1, 2) offered respectively by Maxwell, and Green and Lindsay.

6. Caloric equation

From (19) it follows the equality

(32) $$\frac{d}{dt}(\vartheta \eta) + \frac{d\Psi}{dt} = \frac{de}{dt} + \tau^{ch} \frac{d^2 e}{dt^2} \ .$$

After substituting (17, 19, 29, 30, 31) in (32), for media with central symmetry ($b_k = 0$), we obtain the following non-classic hyperbolic type differential equation describing the propagation of the heat:

(33) $$A\frac{\partial}{\partial t}\theta + B\frac{\partial^2 \theta}{\partial t^2} = \lambda_{kl}\,\theta_{,kl} + \rho(e^{*} + \tau^{ch}\frac{d}{dt}e^{*}) \ .$$

7. Conclusions

In the present study we generalize the Clausius - Duhem inequality by introducing memory functional relationship between the entropy flux and heat flux, which take into account the relaxation of the “chaotic processes”. This basic assumption gives us the ability to formulate a general theory of heat conduction phenomenon, which yields to finite speed of propagation of thermal perturbation and generalization of the heat conduction equations (1,2) offered by Maxwell, and Green and Lindsay.

A principle problem, important for practical applications, is the numerical estimation of the relaxation time τ^{ch} from the experimental data. The studies [22, 23] would be pointed out as rational concern at this direction.

As conclusion we would say that the proposed formulation of the Second Law of thermodynamics could be explored as well in other continuum physical problems beyond the heat conduction.

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