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A Note on the Non-Classical Heat Conduction

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A fundamental drawback of the classical theory of heat conduction [1] is that allows for infinite velocity of propagation of thermal perturbations, which is physically unrealistic. This paradox has first been discussed by Maxwell [2] who generalized the classical Fourier theory [1]. Nowadays similar generalizations have been proposed by Cattaneo [3], Vernotte [4], Luikov [5], etc. They reduce to the following conductive heat flux, h_k, equation:

$$h_k + \tau_r \frac{\partial h_k}{\partial t} = -\lambda_{kl}\theta_{,l} \tag{1}$$

where τ_r is the thermal stress relaxation time and $(.)_{,l}$ denotes differentiation along the coordinate x_l, k, $l=1, 2, 3$, and θ is the thermodynamic temperature.

The infinite velocity of propagation paradoxe has been removed also in the model of thermoelastic media whose thermodynamic behaviour depends on the time rate of change of θ [6], where the heat flux is given by:

$$h_k = b_k \frac{\partial \theta}{\partial t} - \lambda_{kl}\,\theta_{,l} \tag{2}$$

for the case of non-deformable media. In eqs. (1), (2) λ_{kl} and b_k are material constants.

Purpose of the present paper is to derive a rigorous formulation of heat conduction phenomena in non-deformable materials, thus generalizing a number of heat conduction models, that have appeared in literature and also producing some new ones.

Having in mind the general theory of fading memory developed by Coleman, etc. [7—9], the necessary and sufficient conditions to satisfy the I and II principles of thermodynamics in the form of a Clausius-Duhem inequality for non-deformable materials if the validilty of:

$$\eta = -\frac{\partial \widehat{\psi}}{\partial \theta}(A(t),\ H^t(s)) \tag{3}$$

$$\frac{\partial \widehat{\psi}}{\partial \theta_{,k}}(A(t),\ H^t(s)) = 0 \tag{4}$$

$$d\widehat{\psi}((A(t),\ H^t(s)\,|\,\dot{H}^t(s)) - \frac{1}{\theta}\,\theta_{,k}\,h_k \leq 0 \tag{5}$$

where $\widehat{\psi}$ is the free energy, which is a function of the present state of the system $A(t)$ and a functional of the hystory of the system $H^t(s)$;

$$A(t) \overset{\mathrm{Df}}{=} (\theta(t)-\theta_0,\ \theta_{,k}(t))^T \tag{6}$$

$$H^t(s) \overset{\mathrm{Df}}{=} H(t-s) \overset{\mathrm{Df}}{=} (\theta^t(s)-\theta(t),\ \theta^t_{,k}(s))^T ;\quad s \in (0,\ \infty) \tag{7}$$

where $(.,.)^T$ is the transposed quantity.

Eqs. (3), (4) imply that the free energy density $\widehat{\psi}$ and the entropy density η do not depend on the present value of the temperature gradient $\theta_{,k}(t)$. The dependence of the mentioned variables on the temperature gradient's hystory has been argued by Coleman, etc. [10, 11]. They state, that if the hystory of the temperature gradient is considered as an independant variable, the principle of local action would be violated, which is an analog of introducing the second deformation gradient as an independant variable for the case of deformable materials, and hence they neglect it, [8, 9, 12]. It will be shown here that assuming weak non-locality in the above sense enables one to generalize considerably the heat conduction models of non-deformable materials, to formulate a model of the phenomenon mentioned which produces as particular cases all linear heat conduction models, known to the authors presently, and give rise to new ones.

Let the free energy $\widehat{\psi}$ be a quadratic functional of the kind:

$$\varrho_0 \widehat{\psi} \overset{\mathrm{Df}}{=} A + B\,(\theta(t)-\theta_0) + \int_0^\infty F_1^T(s)\,.\,H^t(s)\,ds + \frac{1}{2}\int_0^\infty\int_0^\infty [H^t(s_1)]^T.\ F_2(s_1,\ s_2)\,.\,H^t(s_2)\,ds_1\,ds_2 \tag{8}$$

where

$$F_1^T(s) \overset{\mathrm{Df}}{=} (F(s),\ F_k(s))$$

$$F_2(s_1,\ s_2) \overset{\mathrm{Df}}{=} \begin{bmatrix} K(s_1,\ s_2), & N_l^T(s_1,\ s_2) \\ N_k(s_1,\ s_2) & R_{kl}(s_1,\ s_2) \end{bmatrix}. \tag{9}$$

Eqs. (3)—(9) yield for the entropy density η:

$$-\varrho_0\eta = B + K_0(\theta(t)-\theta_0) + \int_0^\infty K_1(s)\,(\theta(t-s)-\theta_0)ds + \int_0^\infty N_k(s)\,\theta_{,k}(t-s)ds \tag{10}$$

where

$$K_0 = \int_0^\infty\int_0^\infty K(s_1,\ s_2)\,ds_1\,ds_2 \tag{11}$$

$$K_1(s) = -\int_0^\infty K(s,\ s_2)\, ds_2$$

$$N_k(s) = \int_0^\infty N_k(s,\ s_2)\, ds_2.$$

The dissipation inequality, eq. (5), takes the form:

$$\int_0^\infty \left(\frac{d}{ds} F_1(s)\right)^T . \Pi^t(s) ds + \int_0^\infty \int_0^\infty (\Pi^t(s_1))^T . \left(\frac{d}{ds_2} F_2(s_1,\ s_2)\right) . \Pi^t(s_2)\, ds_1\, ds_2 - \theta_{,k}(t)\, \Gamma_k(t) \geqq 0 \tag{12}$$

where

$$\Gamma_k(t) \overset{\mathrm{Df}}{=} \int_0^\infty N_k(s,\ 0)\,(\theta(t-s) - \theta(t)) ds + \int_0^\infty R_{kl}(s,0)\theta_{,l}(t-s) ds - h_k. \tag{13}$$

A necessary and sufficient condition that the dissipation inequality, eq. (12), be satisfied for an arbitrary admissible thermodynamic process is the validity of:

$$\frac{d}{ds} F_1(s) \equiv \mathbf{O};\quad s \in (0,\ \infty)$$

$$\int_0^\infty \int_0^\infty (\Pi^t(s_1))^T . \left(\frac{d}{ds_2} F_2(s_1,\ s_2)\right) . \Pi^t(s_2) ds_1 ds_2 \geqq 0, \tag{14}$$

$$\Gamma_k(t) = \lambda_{kl}\, \theta_{,l}(t),$$

where $\frac{d}{ds_2} F_2(s_1,\ s_2)$ is positive semi-difinite and λ_{kl} — negative semi-difinite.

From eqs. (13) and $(14)_3$ one obtains the following generalized heat flux equation:

$$h_k(t) = -\lambda_{kl}\, \theta_{,l} + \int_0^\infty R_{kl}(s, 0)\, \theta_{,l}(t-s) ds + \int_0^\infty N_k(s,\ 0)\,(\theta(t-s) - \theta(t)) ds. \tag{15}$$

It is eq. (15) that generalizes all linear heat conduction theories known to the authors until this manuscript was prepared. Thus, in the absence of memory effects:

$$R_{kl}(s,\ 0) \equiv 0,\quad N_k(s,\ 0) \equiv 0:\quad s \in (0,\ \infty) \tag{16}$$

the Fourier heat flux equation is obtained. If:

$$R_{kl}(s,\ 0) = \frac{A_{kl}}{\tau_r} e^{-\frac{s}{\tau_r}},\ N_k(s,\ 0) \equiv 0;\quad s \in (0,\ \infty) \tag{17}$$

one has eq. (1) where A_{kl} is material constant tensor. Let:

$$\lambda_{kl} \equiv 0,\quad N_k(s,\ 0) \equiv 0;\quad s \in (0,\ \infty) \tag{18}$$

than from eqs. (15) and (18) one gets:

(19)
$$h_k = \int_0^\infty R_{kl}(s, 0)\, \theta_{,l}(t-s)ds$$

which was proposed by Gurtin and Pipkin [13], while if:

(20)
$$N_k(s, 0) \equiv 0; \quad s \in (0, \infty)$$

from eqs. (15) and (20):

(21)
$$h_k(t) = -\lambda_{kl}\, \theta_{,l} + \int_0^\infty R_{kl}(s, 0)\, \theta_{,l}(t-s)ds$$

formulated and studied by Nunziato [14]. For the case:

(22)
$$R_{kl}(s, 0) \equiv 0; \quad s \in (0, \infty)$$

the heat flux equation is of the form:

(23)
$$h_k = -\lambda_{kl}\, \theta_{,l} + \int_0^\infty N_k(s, 0)\, (\theta(t-s) - \theta(t))ds.$$

Suppose now that the system is characterized by a very strongly fading memory — in this case the kernel $N_k(s, 0)$ might be approximated by the asymmetric Dirak function and its derivative as:

(24)
$$N_k(s, 0) \approx A_k\, \delta_+(s) + B_k \frac{d}{ds}\, \delta_+(s)$$

where A_k and B_k are material constant vectors. Then eqs. (15) and (24) yield the model of Green and Lindsay [6], eq. (2). If like eq. (24), one assumes that:

(25)
$$R_{kl}(s, 0) \sim A_{kl}\delta_+(s) + B_{kl} \frac{d}{ds}\, \delta_+(s)$$

it follows that:

(26)
$$h_k = (\lambda_{kl} + A_{kl})\, \theta_{,l} + B_{kl} \frac{\partial \theta_{,l}}{\partial t} + B_k \frac{\partial \theta}{\partial t}$$

which is not known to the authors from literature.

Thus it can be seen that utilizing the formalism described above, one is able to formulate a general model of heat conduction phenomena, which yields as particular cases a number of known linear heat conduction models and gives rise to new ones too.

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